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Exponential Clogging Time for a One Dimensional DLA

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Abstract In this paper a simple DLA type model is analyzed. In (Benjamini and Yadin in Commun. Math. Phys. 279:187–223, 2008) the standard DLA model from (Witten and Sander in Phys. Rev. B 27:5686–5697, 1983) was considered on a cylinder and the arm growing phenomena was established, provided that the section of the cylinder has sufficiently fast mixing rate. When considering DLA on a cylinder it is natural to ask how many particles it takes to clog the cylinder, e.g. modeling clogging of arteries. In this note we formulate a very simple DLA clogging model and establish an exponential lower bound on the number of particles arriving before clogging appears. In particular we possibly shed some light on why it takes so long to reach the bypass operation.

Keywords Growth process · DLA · Clogging

1 Introduction

We start with an informal description of our model. Fix some $N \in \mathbb{N}$. Initially there is a particle only at the vertices 0 and $1 \in \mathbb{Z}$. A third particle performs a simple random walk started at positive infinity until at some random time when it stops and never moves again. If the particle is at a vertex i, it "freezes" there (stays there and remains there for all time) with probability equal to the number of particles at vertex $i-1$ divided by N. If the particle does not freeze at i then it takes a step in simple random walk. The particle repeats this procedure of either freezing or taking one step of simple random walk until it freezes. When the third particle freezes, the forth particle starts and so on. By the analysis in [1] the cluster of the particles will grow to the right leaving typically only $o(N)$ particles at each site, thus

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it will become harder for particles to penetrate deep beyond the arm to completely fill a site with N particles. Thus the time required before there is one site with N particles is at least exponential in N. This is the main result of this note. N plays the role of the width of the cylinder in this mean field model of clogging a cylinder by diffusing or drifting particles. At the moment we don't know how to prove exponential lower bounds for clogging by DLA on cylinders of the form $G \times \mathbb{Z}$, G finite.

2 Formal Definition of the Process

We will inductively define the following random variables. The variables $\{w(j,t)\}_{j,t\in\mathbb{N}}$ give the location of the jth particle after it has take t steps. The random variables $\{w(j,\infty)\}_{j\in\mathbb{N}}$ indicate where the jth particle freezes. And the variables $\{f(k,i)\}_{k,i\in\mathbb{N}}$ indicate the number of particles labeled less than or equal to i which have frozen at position k.

We define these variables as follows. Let $w(1,t)=0$ for all t. For a fixed j assume we have defined the $w(j,t)$ in such a way that for all $j' \le j$ we have that $\lim_{t\to\infty} w(j',t)$ exists and that we have defined $f(k,i)$ for all $i<j$. Then we define

$$w(j,\infty) = \lim_{t\to\infty} w(j,t)$$

and

$$f(k,i) = |\{i' \le i :\ w(i',\infty) = k\}|.$$

For any $j>1$ let $w(j,1) = j+2$ and $w(j,2) = j+1$.

For any $j>1$ and $t>2$ if $w(j,t) = w(j,t-1)$ then define $w(j,t+1) = w(j,t)$. If $w(j,t) \ne w(j,t-1)$ then set

$$w(j,t+1) = \begin{cases} w(j,t)+1, & \text{with probability } \frac{n-f(w(j,t)-1,j-1)}{2n}; \\ w(j,t)-1, & \text{with probability } \frac{n-f(w(j,t)-1,j-1)}{2n}; \\ w(j,t), & \text{with probability } \frac{f(w(j,t)-1,j-1)}{n}. \end{cases}$$

As simple random walk on $\mathbb{Z}$ is recurrent we have that $w(j,t) = w(j,t+1)$ for some t almost surely. Thus all of the random variables are well defined almost surely.

If $w(j,\infty) = k$ then we say that particle j *freezes* at k. If for some j there exists a t with

$$k = w(j,t) = w(j,t+1) < \min_{t'<t} w(j,t')$$

then $w(j,\infty)=k$ and we say that particle j *freezes upon arrival* at k. If for some k there exists t with $f(k,t) = N$ then we say that there is a *blockage* at k. Define the random variable

$$B = \inf\{k :\ \text{there exists a blockage at } k\}.$$

Thus B indicates the position of the leftmost blockage.

Theorem 2.1 *There exist $c>0$ such that for all N*

$$\mathbf{P}(B < e^{cN}) < e^{-cN}.$$

We make the following comments about our theorem.

- The proof given below clearly works for directed random walks or, more generally, any nearest neighbor process on $\mathbb{Z}$. We only need to modify the model so that the particle either freezes at some location or disappears off to infinity.
- An easy upper bound on B is that for any $\epsilon > 0$ there exists c such that $\mathbf{P}(B > cN^N) < \epsilon$ for all N. It is of interest to get the exact order of B.
- The question from [1] regarding clogging of a cylinder $G \times \mathbb{Z}$ is more complicated due to the geometry of the possible cuts. Finding the distribution on the location of the leftmost clogging in the cylinder is an interesting open question.

3 Proof

For any k let $S(k)$ be the event that there exists i with $f(k,i) = N$.

Lemma 3.1 *There exists $c > 0$ such that for any k and N*

$$\mathbf{P}(S(k)) \leq e^{-cN}.$$

Proof Fix k. We write $\lfloor x \rfloor$ for the largest integer less than or equal to x. Let $I_{N/2}$ ($I_{3N/4}$, $I_{7N/8}$, $I_{15N/16}$) be the minimal value such that $f(k, I_{N/2}) = \lfloor N/2 \rfloor$ ($\lfloor 3N/4 \rfloor$, $\lfloor 7N/8 \rfloor$, $\lfloor 15N/16 \rfloor$ respectively) if such a value exists. If $I_{N/2}$ (or any of $I_{3N/4}$, $I_{7N/8}$, and $I_{15N/16}$) is undefined then $S(k)$ does not occur. Also let

$$g(i) = |\{j \in (I_{N/2}, i) : \text{ particle } j \text{ freezes upon arrival at } k+1\}|.$$

We now consider the following three events:

(1) $g(I_{3N/4}) < 0.24N$,
(2) $g(I_{7N/8}) < 0.59N$, and
(3) $g(I_{15N/16}) < N$.

If $S(k)$ occurs then the third event must occur. To bound the probability that the third event occurs we get

$$\begin{aligned}\mathbf{P}(S(k)) &\leq \mathbf{P}(3)\\ &\leq \mathbf{P}(3) + \mathbf{P}(3|3^C) + \mathbf{P}(3|3^C \cap 3^C)\\ &\leq \mathbf{P}(3) + \mathbf{P}(3|3^C) + \mathbf{P}(3|3^C).\end{aligned}$$

We will now show that all of these probabilities are decreasing exponentially in N.

For every $j \in (I_{N/2}, I_{3N/4}]$ with $\min_t w(j,t) \leq k+1$ the probability that particle j does not freeze upon arrival at $k+1$ is at least one half. If the first event occurs then there is a number $p \leq 0.49N$ such that at least p particles numbered higher that $I_{N/2}$ arrive at $k+1$ and at least $0.25N$ of them do not freeze upon arrival at $k+1$. (If not either $I_{3N/4}$ is undefined or at least $0.24N$ particles with labels in $(I_{N/2}, I_{3N/4}]$ freeze upon arrival at $k+1$. If it is the latter then $g(I_{3N/4}) \geq 0.24N$. Either way the event (3) does not occur.) As $0.25N/0.49N > 0.51$ the probability of this is decreasing exponentially in N.

For every $j \in (I_{3n/4}, I_{7n/8}]$ with $\min_t w(j,t) \leq k+1$ the probability that particle j does not freeze upon arrival at $k+1$ is at most 0.25. If the first event does not occur but the second event does, then there is a number $p \leq 0.475N$ such that at least p particles numbered between $I_{N/2}$ and $I_{3N/4}$ arrive at $k+1$ and at least $0.125N$ of them do not freeze upon arrival at $k+1$. (If not either $I_{7N/8}$ is undefined or at least $0.35N$ particles with labels in $(I_{3N/4}, I_{7N/8}]$ freeze upon arrival at $k+1$. If it is the latter then $g(I_{7N/8}) \geq g(I_{3N/4}) + 0.35N \geq 0.59N$. Either way the event (3) does not occur.) As $0.125N/0.475N > 0.26$ the conditional probability of this given the compliment of the first event is decreasing exponentially in N.

For every $j \in (I_{7N/8}, I_{15N/16}]$ with $\min_t w(j,t) \leq k+1$ the probability that the particle does not freeze upon arrival at $k+1$ is at most 0.125. If the second event does not occur but the third event does, then there is a number $p \leq 0.4725N$ such that at least p particles numbered between $I_{7N/8}$ and $I_{15N/16}$ arrive at $k+1$ and at least $0.0625N$ of them do not freeze upon arrival at $k+1$. (If not either $I_{15N/16}$ is undefined or at least $0.41N$ particles with labels in $(I_{7N/8}, I_{15N/16}]$ freeze upon arrival at $k+1$. If it is the latter then $g(I_{15N/16}) \geq g(I_{7N/8}) + 0.41N \geq N$. Either way the event (3) does not occur.) As $0.0625N/0.4725N > 0.13$ the conditional probability of the third event given the compliment of the second event is decreasing exponentially in N.

Thus the probability that there exists i such that $C(k, I_{N/2}, i) \geq N/2$ and $g(i) < N$ is exponentially small in N. Thus the probability of $S(k)$ is as well. □

Proof of Theorem 2.1 This follows by replacing c with $c/2$ in Lemma 3.1. □

References

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